

A TWISTED FIRST HOMOLOGY GROUP OF THE GOERITZ GROUP OF S^3

AKIRA KANADA

ABSTRACT. Given a genus- g Heegaard splitting of a 3-sphere, the genus- g Goeritz group \mathcal{E}_g is defined to be the group of the isotopy classes of orientation preserving homeomorphism of the 3-sphere that preserve the splitting. In this paper, we determine the twisted first (co)homology group of the genus-2 Goeritz group of 3-sphere.

1. INTRODUCTION

Mapping class group. Let H_g be a 3-dimensional handlebody of genus g , and Σ_g be the boundary surface ∂H_g . We denote by \mathcal{M}_g the mapping class group of Σ_g , the group of isotopy classes of orientation preserving homeomorphisms of Σ_g . Dehn [2] proved that \mathcal{M}_g is generated by finitely many Dehn twists. Furthermore Lickorish [19, 20] proved that $3g - 1$ Dehn twists generate \mathcal{M}_g . Humphries [16] found that $2g + 1$ Dehn twists generate \mathcal{M}_g . We denote by \mathcal{H}_g the handlebody mapping class group, the subgroup of mapping class group \mathcal{M}_g of boundary surface ∂H_g defined by isotopy classes of those orientation preserving homeomorphisms of ∂H_g which can be extended to homeomorphisms of H_g . It turns out that \mathcal{H}_g can be identified with the group of isotopy classes of orientation preserving homeomorphisms of H_g . A finite presentation of the handlebody mapping class group \mathcal{H}_g was obtained by Wajnryb [1].

Goeritz group. Let H_g and H_g^* be 3-dimensional handlebodies, and $M = H_g \cup H_g^*$ be a Heegaard splitting of a closed orientable 3-manifold M . Let \mathcal{M}_g be the mapping class group of the boundary surface $\partial H_g = \Sigma_g$. The group of mapping classes $[f] \in \mathcal{M}_g$ such that there is an orientation preserving self-homeomorphism F of (M, H_g) satisfying $[F|_{\partial H_g}] = [f]$ is called the genus- g Goeritz group of $M = H_g \cup H_g^*$. When a manifold M admits a unique Heegaard splitting of genus g up to isotopy, we can define the genus- g Goeritz group of the manifold without mentioning a specific splitting. For example, the 3-sphere, $S^1 \times S^2$ and lens spaces are known to be such manifolds from [6], [4] and [5].

In studying Goeritz groups, finding their generating sets or presentations has been an interesting problem. However the generating sets or the presentation of those groups have been obtained only for a few manifolds with their splittings of small genera. A finite presentation of the genus-2 Goeritz group of 3-sphere was obtained [3]. In an arbitrary genus, first Powell [8] and then Hirose [17] claimed that they have found a finite generating set for the genus- g Goeritz group of 3-sphere, though serious gaps in both arguments were found by Scharlemann. Establishing the existence of such generating sets appears to be an open problem.

In addition, finite presentations of the genus-2 Goeritz groups of each lens spaces $L(p, 1)$ were obtained [12], other lens spaces were obtained [15] and the genus-2 Heegaard splittings of non-prime 3-manifolds were obtained [14]. Recently a finite presentation of the genus-2 Goeritz group of $S^1 \times S^2$ was obtained [13].

Homology of mapping class group. Computing homology of mapping class groups is interesting topic of studying mapping class groups. Harer [9] determined the second homology group of mapping class group \mathcal{M}_g :

$$H_2(\mathcal{M}_g; \mathbb{Z}) \cong \mathbb{Z} \text{ if } g \geq 4.$$

In fact, Harer proved a more general theorem for surfaces with multiple boundary components and arbitrarily many punctures.

In twisted case, Morita [18] determined the first homology group with coefficients in the first integral homology group of the surface:

$$H_1(\mathcal{M}_g; H_1(\Sigma_g)) \cong \mathbb{Z}/(2g-2)\mathbb{Z} \text{ if } g \geq 2.$$

Recently Ishida and Sato [7] computed the twisted first homology groups of the handlebody mapping class group \mathcal{H}_g with coefficients in the first integral homology group of the boundary surface Σ_g :

$$H_1(\mathcal{H}_g; H_1(\Sigma_g)) \cong \begin{cases} \mathbb{Z}/(2g-2)\mathbb{Z} & \text{if } g \geq 4, \\ \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z} & \text{if } g = 3, \\ (\mathbb{Z}/2\mathbb{Z})^2 & \text{if } g = 2. \end{cases}$$

Goeritz group of S^3 . Let H_g and H_g^* be 3-dimensional handlebodies, and $S^3 = H_g \cup H_g^*$ be the Heegaard splitting of the 3-sphere S^3 . Waldhausen [6] proved that a genus- g Heegaard splitting of S^3 is unique up to isotopy. Let \mathcal{M}_g be the mapping class group of the boundary surface $\partial H_g = \Sigma_g$. The group of mapping classes $[f] \in \mathcal{M}_g$ such that there is an orientation preserving self-homeomorphism F of (S^3, H_g) satisfying $[F|_{\partial H_g}] = [f]$ is denoted by \mathcal{E}_g . It is called the genus- g Goeritz group of the 3-sphere.

Twisted homology group of \mathcal{E}_2 . In this paper, we compute the twisted first homology group of \mathcal{E}_2 with coefficients in the first integral homology group of the Heegaard surface Σ_2 . The following is the main theorem in this paper.

Theorem 1.1.

$$H_1(\mathcal{E}_2; H_1(\Sigma_2)) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

A finite presentation of the genus-2 Goeritz group of the 3-sphere was obtained from the works of [3]. For the higher genus Goeritz groups of the 3-sphere, it is conjectured that all of them are finitely presented however it is still known to be an open problem.

Let Σ_g be a compact connected orientable surface of genus $g \geq 1$ and $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ be oriented simple closed curves as in Figure 1. We denote their homology classes in $H_1(\Sigma_g)$ by $x_1 = [\alpha_1], x_2 = [\alpha_2], \dots, x_g = [\alpha_g], y_1 = [\beta_1], y_2 = [\beta_2], \dots, y_g = [\beta_g]$. The basis $\{x_1, \dots, x_g, y_1, \dots, y_g\}$ of H_A induces an isomorphism $H_A \simeq A^{2g}$. For $v \in A^{2g}$, we denote its projection to the i -th coordinate of A^{2g} by v_i for $i = 1, 2, \dots, 2g$.

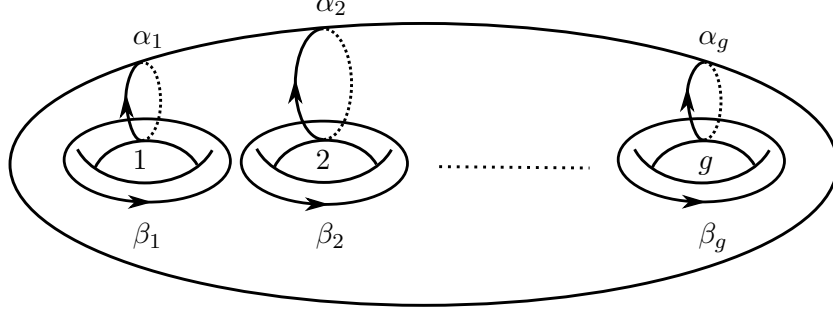


Figure 1 : Surface Σ_g and simple closed curves $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$.

Akbas gave following presentation for \mathcal{E}_2 in [3].

Theorem 1.2 ([3]). *The group \mathcal{E}_2 has four generators $[\alpha], [\beta], [\gamma]$ and $[\delta]$, and the following relations:*

- (P1) $[\alpha]^2 = [\beta]^2 = [\delta]^3 = [\alpha\gamma]^2 = 0$.
- (P2) $[\alpha\delta\alpha] = [\delta]$ and $[\alpha\beta\alpha] = [\beta]$.
- (P3) $[\gamma\beta\gamma] = [\alpha\beta]$ and $[\delta] = [\gamma\delta^2\gamma]$.

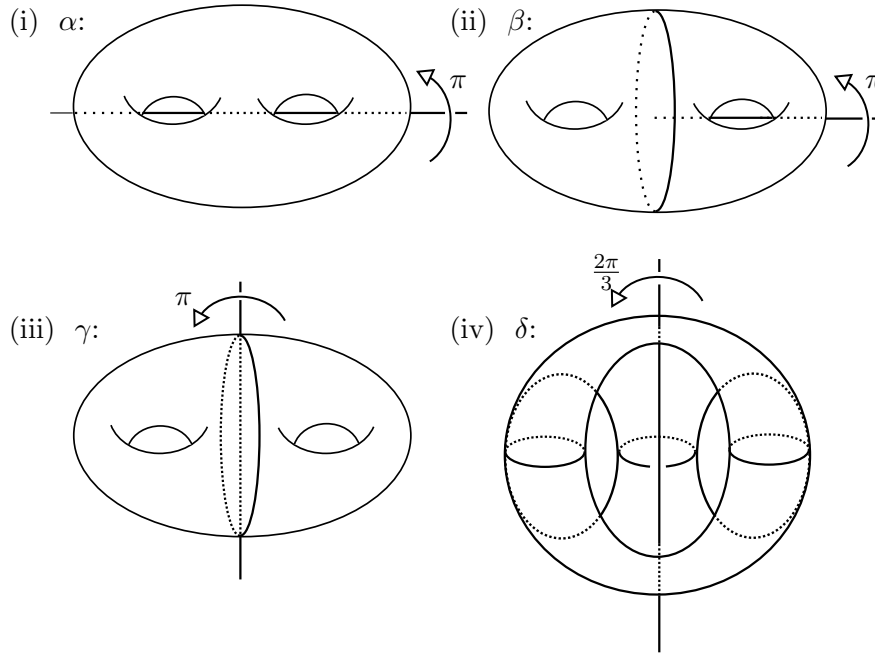
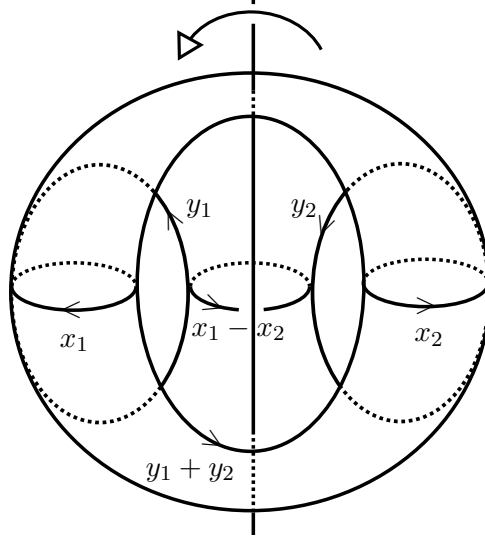


Figure 2 : Generators of \mathcal{E}_2 .

We define δ as follows. Consider the genus-two handlebody as a regular neighborhood of a sphere, centered at the origin, with three holes. The homeomorphism δ is a $2\pi/3$ rotation of the handlebody about the vertical z -axis. See Figure 2. Scharlemann [11] showed that the group \mathcal{E}_2 is generated by isotopy classes $[\alpha]$, $[\beta]$, $[\gamma]$ and $[\delta]$. Correspondence of homology classes of (iv) and the others are as follows:



2. TWISTED COHOMOLOGY GROUP OF \mathcal{E}_2

We denote by A the ring \mathbb{Z} or $\mathbb{Z}/n\mathbb{Z}$ for an integer $n \geq 2$, and set $H_A = H_1(\Sigma_g; A)$. For a group G and a left G -module N , a map $d : G \rightarrow N$ is called a *crossed homomorphism* if it satisfies $d(gg') = d(g) + gd(g')$ for $g, g' \in G$. Now let $Z^1(\mathcal{E}_g; H_A)$ be the set of all crossed homomorphisms $d : \mathcal{E}_g \rightarrow H_A$. Namely

$$Z^1(\mathcal{E}_g; H_A) = \{d : \mathcal{E}_g \rightarrow H_A; d(\phi\psi) = d(\phi) + \phi d(\psi), \phi, \psi \in \mathcal{E}_g\}.$$

Let $\pi : H_A \rightarrow Z^1(\mathcal{E}_g; H_A)$ be the homomorphism defined by

$$\pi(u)(\phi) = \phi u - u$$

for $u \in H_A$. Then as is well known we have

$$H^1(\mathcal{E}_g; H_A) = Z^1(\mathcal{E}_g; H_A) / \text{Im } \pi$$

(cf. K.S. Brown [10]).

We consider the case $g = 2$. Then we have the homomorphism $\mathcal{E}_2 \rightarrow \text{Aut}(H_1(\Sigma_2; \mathbb{Z}))$ induced by the action of the group \mathcal{E}_2 on $H_1(\Sigma_2; \mathbb{Z})$. The action of α, β, γ , and δ is as follows:

$$\begin{aligned} \alpha_*: & \alpha_*(x_i) = -x_i \text{ and } \alpha_*(y_i) = -y_i \ (i = 1, 2). \\ \beta_*: & \beta_*(x_1) = x_1, \beta_*(x_2) = -x_2, \beta_*(y_1) = y_1, \beta_*(y_2) = -y_2. \\ \gamma_*: & \gamma_*(x_1) = -x_2, \gamma_*(x_2) = -x_1, \gamma_*(y_1) = -y_2, \gamma_*(y_2) = -y_1. \\ \delta_*: & \delta_*(x_1) = -x_1 + x_2, \delta_*(x_2) = -x_1, \delta_*(y_1) = y_2, \delta_*(y_2) = -y_1 - y_2. \end{aligned}$$

For a group G and a left G -module N , the *coinvariant* N_G is quotient module of N by the subgroup $\{gn - n | g \in G, n \in N\}$.

Lemma 2.1.

$$H_1(\Sigma_2)_{\mathcal{E}_2} = 0.$$

Proof. Since we have $\alpha_*\delta_*^2(-x_2) = x_1 - x_2$ and $\gamma_*(x_1) = -x_2$, we obtain $x_1 = x_2 = 0 \in H_1(\Sigma_2)_{\mathcal{E}_2}$. And we have $\gamma_*\delta_*^2(y_1) = y_1 + y_2$ and $\gamma_*(y_1) = -y_2$. Hence we also obtain $y_1 = y_2 = 0 \in H_1(\Sigma_2)_{\mathcal{E}_2}$. \square

Lemma 2.2. Let G_i , H_i and K be G -modules ($i = 1, 2, 3$), and let

$$\begin{aligned} \cdots \rightarrow G_3 \rightarrow G_2 \rightarrow K \rightarrow G_1 \rightarrow 0 \text{ and} \\ 0 \rightarrow H_1 \rightarrow K \rightarrow H_2 \rightarrow H_3 \rightarrow \cdots \end{aligned}$$

be exact sequences. If $G_2 \rightarrow K \rightarrow H_2$ is an isomorphism, then we have $H_1 \rightarrow K \rightarrow G_1$ is isomorphism.

Proof. Now we have the following diagram. Set $\Phi = g_1 \circ f_1$ and $\Psi = f_2 \circ g_2$.

$$\begin{array}{ccccccc} & & & & & & 0 \\ & & & & & & \swarrow \\ & & & & & H_1 & \\ & & & & f_1 \swarrow & \downarrow & \\ \cdots & \longrightarrow & G_3 & \xrightarrow{g_3} & G_2 & \xrightarrow{g_2} & K \xrightarrow{g_1} G_1 \longrightarrow 0 \\ & & & & \downarrow & \swarrow f_2 & \\ & & & & H_2 & & \\ & & & & \swarrow f_3 & & \\ & & & & H_3 & & \\ & & & & \swarrow & & \\ & & & & \cdots & & \end{array}$$

Note that g_2 is injective and f_2 is surjective if $G_2 \rightarrow K \rightarrow H_2$ is an isomorphism. Thus we have $g_3 = 0$ and $f_3 = 0$. Since Ψ is isomorphism, we have $(\Psi^{-1} \circ f_2) \circ g_2 = id_{G_2}$ and $f_2 \circ (g_2 \circ \Psi^{-1}) = id_{H_2}$. Hence those exact sequences are split and we have

$$\begin{aligned} H_1 \oplus H_2 &\xrightarrow{\cong} K : (h_1, h_2) \mapsto f_1(h_1) + g_2 \circ \Psi^{-1}(h_2), \\ K &\xrightarrow{\cong} G_1 \oplus G_2 : k \mapsto (g_1(k), \Psi^{-1} \circ f_2(k)). \end{aligned}$$

A composition map $H_1 \oplus H_2 \rightarrow K \rightarrow G_1 \oplus G_2$ is

$$\begin{aligned} (h_1, h_2) &\mapsto (g_1(f_1(h_1) + g_2 \circ \Psi^{-1}(h_2)), \Psi^{-1} \circ f_2(f_1(h_1) + g_2 \circ \Psi^{-1}(h_2))) \\ &= (g_1 \circ f_1(h_1) + g_1 \circ g_2 \circ \Psi^{-1}(h_2), \Psi^{-1} \circ f_2 \circ f_1(h_1) + \Psi^{-1} \circ f_2 \circ g_2 \circ \Psi^{-1}(h_2)) \\ &= (\Phi(h_1), \Psi^{-1}(h_2)). \end{aligned}$$

Hence Ψ is an isomorphism. \square

Lemma 2.3. *In the case $g = 2$, we have*

$$H^1(\mathcal{E}_2; H_A) \cong \{d \in Z^1(\mathcal{E}_2; H_A); \\ d([\delta])_1 - d([\alpha])_1 = d([\gamma])_2 - d([\beta])_2 = d([\gamma])_3 - d([\alpha])_3 = d([\beta])_4 - d([\delta])_4 = 0\}.$$

Proof. Let $f : Z^1(\mathcal{E}_2; H_A) \rightarrow A^4$ be a homomorphism defined by

$$f(d) = (d([\delta])_1 - d([\alpha])_1, d([\gamma])_2 - d([\beta])_2, d([\gamma])_3 - d([\alpha])_3, d([\beta])_4 - d([\delta])_4).$$

Since we have

$$\begin{aligned} \alpha v - v &= (-2v_1, -2v_2, -2v_3, -2v_4), \\ \beta v - v &= (0, -2v_2, 0, -2v_4), \\ \gamma v - v &= (-v_1 - v_2, -v_1 - v_2, -v_3 - v_4, -v_3 - v_4), \\ \delta v - v &= (-2v_1 + v_2, -v_1 - v_2, -v_3 + v_4, -v_3 - 2v_4), \end{aligned}$$

the composition map $f \circ \pi : H_A \rightarrow A^4$ is written as

$$f \circ \pi(v) = (v_2, -v_1 + v_2, v_3 - v_4, -v_3)$$

for $v \in H_A$. This map is an isomorphism. We have the following diagram.

$$\begin{array}{ccccccc} & & & & \text{Ker } f & \longleftarrow & 0 \\ & & & & \swarrow & & \downarrow \\ 0 & \longrightarrow & H_A & \xrightarrow{\pi} & Z^1(\mathcal{E}_2; H_A) & \longrightarrow & \text{Coker } \pi \longrightarrow 0 \\ & & \downarrow \cong & & \swarrow f & & \\ 0 & \longleftarrow & A^4 & & & & \end{array}$$

By Lemma 2.2, we have

$$H^1(\mathcal{E}_2; H_A) = Z^1(\mathcal{E}_2; H_A) / \text{Im } \pi \cong \text{Ker } f.$$

□

The group \mathcal{E}_2 is generated by $[\alpha]$, $[\beta]$, $[\gamma]$, and $[\delta]$. Therefore, all crossed homomorphisms $d : \mathcal{E}_2 \rightarrow H_A$ are determined by the values $d([\alpha])$, $d([\beta])$, $d([\gamma])$ and $d([\delta])$. If $d \in Z^1(\mathcal{E}_2, H_A)$, we can set

$$\begin{aligned} d([\alpha]) &= \omega_{11}x_1 + \omega_{12}x_2 + \omega_{13}y_1 + \omega_{14}y_2, \\ d([\beta]) &= \omega_{21}x_1 + \omega_{22}x_2 + \omega_{23}y_1 + \omega_{24}y_2, \\ d([\gamma]) &= \omega_{31}x_1 + \omega_{32}x_2 + \omega_{33}y_1 + \omega_{34}y_2, \\ d([\delta]) &= \omega_{41}x_1 + \omega_{42}x_2 + \omega_{43}y_1 + \omega_{44}y_2. \end{aligned}$$

Then we have

$$H^1(\mathcal{E}_2; H_A) \cong \{d \in Z^1(\mathcal{E}_2; H_A) ; \omega_{41} - \omega_{11} = \omega_{32} - \omega_{22} = \omega_{33} - \omega_{13} = \omega_{24} - \omega_{44} = 0\}.$$

3. RELATIONS OF ω_{ij}

In this section we shall consider the case $g = 2$. We denote $d([\phi])$ for $\phi \in \mathcal{E}_2$ simply by $d(\phi)$.

Lemma 3.1. *We have relations:*

$$\begin{cases} \omega_{11} + \omega_{12} = \omega_{31} + \omega_{32}, & (1a) \\ \omega_{13} + \omega_{14} = \omega_{33} + \omega_{34}. & (1b) \end{cases}$$

Proof. By the relations $(\alpha\gamma)^2 = 0$ in (P1), we have $d((\alpha\gamma)^2) = 0$. The equation

$$\begin{aligned} d((\alpha\gamma)^2) &= d(\alpha\gamma) + \alpha\gamma d(\alpha\gamma) \\ &= d(\alpha) + \alpha d(\alpha) + \alpha\gamma d(\alpha) + \alpha\gamma\alpha d(\gamma) \\ &= \omega_{11}x_1 + \omega_{12}x_2 + \omega_{13}y_1 + \omega_{14}y_2 + (-\omega_{31}x_1 - \omega_{32}x_2 - \omega_{33}y_1 - \omega_{34}y_2) \\ &\quad + \omega_{12}x_1 + \omega_{11}x_2 + \omega_{14}y_1 + \omega_{13}y_2 + (-\omega_{32}x_1 - \omega_{31}x_2 - \omega_{34}y_1 - \omega_{33}y_2) \\ &= (\omega_{11} + \omega_{12} - \omega_{31} - \omega_{32})x_1 + (\omega_{11} + \omega_{12} - \omega_{31} - \omega_{32})x_2 \\ &\quad + (\omega_{13} + \omega_{14} - \omega_{33} - \omega_{34})y_1 + (\omega_{13} + \omega_{14} - \omega_{33} - \omega_{34})y_2 \\ &= 0 \end{aligned}$$

holds. Hence, we obtain (1a) and (1b). □

Lemma 3.2. *We have relations:*

$$\begin{cases} 2\omega_{11} + \omega_{12} = 2\omega_{41}, & (2a) \\ -\omega_{11} + \omega_{12} = 2\omega_{42}, & (2b) \\ \omega_{13} + \omega_{14} = 2\omega_{43}, & (2c) \\ -\omega_{13} + 2\omega_{14} = 2\omega_{44}, & (2d) \end{cases}$$

Proof. By the relations $\alpha\delta\alpha = \delta$ in (P2), we have $d(\alpha\delta\alpha) = d(\delta)$ and

$$\begin{aligned} d(\alpha\delta\alpha) &= d(\alpha) + \alpha d(\delta) + \alpha\delta d(\alpha) \\ &= \omega_{11}x_1 + \omega_{12}x_2 + \omega_{13}y_1 + \omega_{14}y_2 + (-\omega_{41}x_1 - \omega_{42}x_2 - \omega_{43}y_1 - \omega_{44}y_2) \\ &\quad + \omega_{11}(x_1 - x_2) + \omega_{12}x_1 - \omega_{13}y_2 + \omega_{14}(y_1 + y_2) \\ &= (2\omega_{11} - \omega_{41} + \omega_{12})x_1 + (\omega_{12} - \omega_{42} - \omega_{11})x_2 \\ &\quad + (\omega_{13} - \omega_{43} + \omega_{14})y_2 + (2\omega_{14} - \omega_{44} - \omega_{13})y_2, \\ d(\delta) &= \omega_{41}x_1 + \omega_{42}x_2 + \omega_{43}y_1 + \omega_{44}y_2. \end{aligned}$$

Comparing $d(\alpha\delta\alpha)$ and $d(\delta)$, we obtain (2a) – (2d). □

Lemma 3.3. *We have relations:*

$$\begin{cases} 2\omega_{21} = 2\omega_{23} = 0, & (3a) \\ 2\omega_{12} = 2\omega_{22}, & (3b) \\ 2\omega_{14} = 2\omega_{24}. & (3c) \end{cases}$$

Proof. By the relation $\alpha\beta\alpha = \beta$ in (P2), we have $d(\alpha\beta\alpha) = d(\beta)$ and

$$\begin{aligned}
d(\alpha\beta\alpha) &= d(\alpha) + \alpha d(\beta\alpha) \\
&= d(\alpha) + \alpha d(\beta) + \alpha\beta d(\alpha) \\
&= \omega_{11}x_1 + \omega_{12}x_2 + \omega_{13}y_1 + \omega_{14}y_2 \\
&\quad - \omega_{21}x_1 - \omega_{22}x_2 - \omega_{23}y_1 - \omega_{24}y_2 \\
&\quad - \omega_{11}x_1 + \omega_{12}x_2 - \omega_{13}y_1 + \omega_{14}y_2 \\
&= -\omega_{21}x_1 + (2\omega_{12} - \omega_{22})x_2 - \omega_{23}y_1 + (2\omega_{14} - \omega_{24})y_2, \\
d(\beta) &= \omega_{21}x_1 + \omega_{22}x_2 + \omega_{23}y_2 + \omega_{24}y_2.
\end{aligned}$$

Comparing $d(\alpha\beta\alpha)$ and $d(\beta)$, we obtain (3a) – (3c). □

Lemma 3.4. *We have relations:*

$$\begin{cases} \omega_{31} - \omega_{32} - \omega_{22} = \omega_{11} + \omega_{21}, & (4a) \\ \omega_{32} + \omega_{31} - \omega_{21} = \omega_{12} - \omega_{22}, & (4b) \\ \omega_{33} - \omega_{34} - \omega_{24} = \omega_{13} + \omega_{23}, & (4c) \\ \omega_{34} + \omega_{33} - \omega_{23} = \omega_{14} - \omega_{24}. & (4d) \end{cases}$$

Proof. By the relation $\gamma\beta\gamma = \alpha\beta$ in (P3), we have $d(\gamma\beta\gamma) = d(\alpha\beta)$ and

$$\begin{aligned}
d(\gamma\beta\gamma) &= d(\gamma) + \gamma d(\beta\gamma) \\
&= d(\gamma) + \gamma d(\beta) + \gamma\beta d(\gamma) \\
&= \omega_{31}x_1 + \omega_{32}x_2 + \omega_{33}y_1 + \omega_{34}y_2 \\
&\quad - \omega_{21}x_2 - \omega_{22}x_1 - \omega_{23}y_2 - \omega_{24}y_1 \\
&\quad + \omega_{31}x_2 - \omega_{32}x_1 + \omega_{33}y_2 - \omega_{34}y_1 \\
&= (\omega_{31} - \omega_{32} - \omega_{22})x_1 + (\omega_{31} + \omega_{32} - \omega_{21})x_2 \\
&\quad + (\omega_{33} - \omega_{34} - \omega_{24})y_1 + (\omega_{33} + \omega_{34} - \omega_{23})y_2, \\
d(\alpha\beta) &= d(\alpha) + \alpha d(\beta) \\
&= (\omega_{11} + \omega_{21})x_1 + (\omega_{12} - \omega_{22})x_2 + (\omega_{13} - \omega_{23})y_1 + (\omega_{14} + \omega_{24})y_2.
\end{aligned}$$

Comparing $d(\gamma\beta\gamma)$ and $d(\alpha\beta)$, we obtain (4a) – (4d). □

Lemma 3.5. *We have relations:*

$$\begin{cases} 2\omega_{31} + \omega_{32} = 2\omega_{41} + \omega_{42}, & (5a) \\ \omega_{33} = \omega_{43}. & (5b) \end{cases}$$

Proof. By the relation $\gamma\delta^2\gamma = \delta$ in (P3), we have $d(\gamma\delta^2\gamma) = d(\delta)$ and

$$\begin{aligned}
d(\gamma\delta^2\gamma) &= d(\gamma) + \gamma d(\delta^2\gamma) \\
&= d(\gamma) + \gamma d(\delta) + \gamma\delta d(\delta) + \gamma\delta^2 d(\gamma) \\
&= \omega_{31}x_1 + \omega_{32}x_2 + \omega_{33}y_1 + \omega_{34}y_2 \\
&\quad - \omega_{42}x_1 - \omega_{41}x_2 - \omega_{44}y_1 - \omega_{43}y_2 \\
&\quad - \omega_{41}x_1 + (\omega_{41} + \omega_{42})x_2 + (-\omega_{43} + \omega_{44})y_1 + \omega_{44}y_2 \\
&\quad + (\omega_{31} + \omega_{32})x_1 - \omega_{32}x_2 + \omega_{33}y_1 + (\omega_{33} - \omega_{34})y_2 \\
&= (2\omega_{31} - \omega_{41} - \omega_{42} + \omega_{32})x_1 + \omega_{42}x_2 \\
&\quad + (2\omega_{33} - \omega_{43})y_1 + (\omega_{33} + \omega_{44} - \omega_{43})y_2, \\
d(\delta) &= \omega_{41}x_1 + \omega_{42}x_2 + \omega_{43}y_2 + \omega_{44}y_2.
\end{aligned}$$

Comparing $d(\gamma\delta^2\gamma)$ and $d(\delta)$, we obtain (5a) and (5b). \square

By Lemma 3.1, 3.2, 3.3, 3.4 and 3.5, we obtain the following equations.

$$\begin{cases} \omega_{11} + \omega_{12} = \omega_{31} + \omega_{32}, & (1a) \\ \omega_{13} + \omega_{14} = \omega_{33} + \omega_{34}, & (1b) \end{cases} \quad \begin{cases} 2\omega_{11} + \omega_{12} = 2\omega_{41}, & (2a) \\ -\omega_{11} + \omega_{12} = 2\omega_{42}, & (2b) \\ \omega_{13} + \omega_{14} = 2\omega_{43}, & (2c) \\ -\omega_{13} + 2\omega_{14} = 2\omega_{44}, & (2d) \end{cases}$$

$$\begin{cases} 2\omega_{21} = 2\omega_{23} = 0, & (3a) \\ 2\omega_{12} = 2\omega_{22}, & (3b) \\ 2\omega_{14} = 2\omega_{24}, & (3c) \end{cases} \quad \begin{cases} \omega_{31} - \omega_{32} - \omega_{22} = \omega_{11} + \omega_{21}, & (4a) \\ \omega_{31} + \omega_{32} - \omega_{21} = \omega_{12} - \omega_{22}, & (4b) \\ \omega_{33} - \omega_{34} - \omega_{24} = \omega_{13} + \omega_{23}, & (4b) \\ \omega_{34} + \omega_{33} - \omega_{23} = \omega_{14} - \omega_{24}, & (4b) \end{cases}$$

$$\begin{cases} 2\omega_{31} + \omega_{32} = 2\omega_{41} + \omega_{42}, & (5a) \\ \omega_{33} = \omega_{43}. & (5b) \end{cases}$$

4. CALCULATION OF COHOMOLOGY

In this section we prove that $H^1(\mathcal{E}_2; H_A) \cong \text{Hom}((\mathbb{Z}/2\mathbb{Z})^2, A)$. The universal coefficient theorem implies $H_1(\mathcal{E}_2; H_1(\Sigma_2)) \cong (\mathbb{Z}/2\mathbb{Z})^2$. To determine the twisted first cohomology group of \mathcal{E}_2 , we solve equations (1a) – (5b) and the condition $d \in \text{Ker } f$, i.e.

$$(*) \quad \omega_{41} - \omega_{11} = \omega_{32} - \omega_{22} = \omega_{33} - \omega_{13} = \omega_{24} - \omega_{44} = 0.$$

Lemma 4.1. *We have a relation of $\text{Ker } f$:*

$$\omega_{12} = 0.$$

Proof. Using (2a) and $\omega_{11} = \omega_{41}$ by (*), we obtain

$$\omega_{12} = 0.$$

\square

Lemma 4.2. *The elements ω_{21} , ω_{22} , ω_{31} , ω_{32} and ω_{42} have order 2 and*

$$\omega_{22} = \omega_{31} = \omega_{32} = -\omega_{21} = -\omega_{42}.$$

Proof. By (2a) and (2b), we have $\omega_{11} + 2\omega_{12} = 2\omega_{41} + 2\omega_{42}$. By (5a) we have $2\omega_{41} + 2\omega_{42} = 2\omega_{31} + \omega_{32} + \omega_{42}$. Using these two equations and (1a), we have $\omega_{12} = \omega_{31} + \omega_{42}$. Since $\omega_{12} = 0$, the equation

$$\omega_{31} + \omega_{42} = 0 \quad (4.2.1)$$

holds. Using (2b) and Lemma 4.1, we obtain

$$\omega_{11} + 2\omega_{42} = 0. \quad (4.2.2)$$

The equation

$$\begin{aligned} \omega_{31} + \omega_{32} &\stackrel{(1a)}{=} \omega_{11} + \omega_{12} \\ &\stackrel{(Lem4.1)}{=} \omega_{11} \\ &\stackrel{(4.2.2)}{=} -2\omega_{42} \\ &\stackrel{(4.2.1)}{=} 2\omega_{31} \end{aligned}$$

holds. So we obtain

$$\omega_{31} - \omega_{32} = 0. \quad (4.2.3)$$

We have

$$\begin{aligned} \omega_{21} &\stackrel{(4a)}{=} \omega_{31} - \omega_{32} - \omega_{11} - \omega_{22} \\ &\stackrel{(4.2.3)}{=} -\omega_{11} - \omega_{22} \\ &\stackrel{(**)}{=} -2\omega_{22} - \omega_{22} \\ &= -3\omega_{22} \\ &\stackrel{(3b)}{=} -\omega_{22} - 2\omega_{12} \\ &\stackrel{(Lem4.1)}{=} -\omega_{22}. \end{aligned}$$

Here, (**) is the equation $\omega_{11} = 2\omega_{22}$. This equation is obtained as follows:

$$\omega_{11} \stackrel{(4.2.2)}{=} -\omega_{42} \stackrel{(4.2.1)}{=} 2\omega_{31} \stackrel{(4.2.3)}{=} 2\omega_{32} \stackrel{(*)}{=} 2\omega_{22}.$$

By (3b) and Lemma4.1, we have $-2\omega_{22} = 0$. Hence we obtain

$$2\omega_{21} = -2\omega_{22} = 0.$$

□

From the results of Lemma 4.2 and (4.2.2), we have

$$\omega_{11} \stackrel{(*)}{=} \omega_{41} = 0.$$

Lemma 4.3. *We have a relation:*

$$\omega_{13} = \omega_{14} = \omega_{33} = \omega_{34} = \omega_{43} = 0.$$

Proof. By (5b) and (*), we obtain

$$\omega_{13} = \omega_{33} = \omega_{43} = \omega_{14}.$$

Using this equation and (1b), we have $\omega_{14} = \omega_{34}$. So the equation

$$\omega_{13} = \omega_{33} = \omega_{43} = \omega_{14} = \omega_{34} \quad (4.3.1)$$

holds. The equation

$$\begin{aligned} 2\omega_{44} &\stackrel{(2d)}{=} 2\omega_{14} - \omega_{13} \\ &\stackrel{(4.3.1)}{=} \omega_{14} \end{aligned}$$

holds. Hence we have

$$4\omega_{44} = 2\omega_{14} \stackrel{(3c)}{=} 2\omega_{24} \stackrel{(*)}{=} 2\omega_{44}.$$

So we obtain

$$2\omega_{44} = 0.$$

Since the equation $2\omega_{44} = \omega_{14}$ holds, we complete the proof of Lemma 4.3. \square

Lemma 4.4. *The elements ω_{23} , ω_{24} and ω_{44} have order 2 and are equal to each other:*

$$\omega_{23} = \omega_{24} = \omega_{44}, \quad 2\omega_{23} = 0.$$

Proof. By (3a), the element ω_{23} have order 2. Since the equation

$$\begin{aligned} \omega_{24} &\stackrel{(4b)}{=} \omega_{14} - \omega_{33} - \omega_{34} + \omega_{23} \\ &\stackrel{(Lem4.3)}{=} \omega_{23} \end{aligned}$$

holds, we obtain

$$\omega_{23} = \omega_{24} \stackrel{(*)}{=} \omega_{44}.$$

\square

Proof of Theorem 1.1. From Lemmas 3,7, 4.1, 4.2, 4.3 and 4.4, we have

$$\begin{aligned} d(\alpha) &= 0, \\ d(\beta) &= \omega_{22}(-x_1 + x_2) + \omega_{23}(y_1 + y_2), \\ d(\gamma) &= \omega_{22}(x_1 + x_2), \\ d(\delta) &= \omega_{42}x_2 + \omega_{23}y_2, \end{aligned}$$

where $-\omega_{22} = \omega_{42}$ and $2\omega_{22} = 2\omega_{23} = 0$. Hence it follows

$$H^1(\mathcal{E}_2; H_A) \cong \text{Ker } f \cong \{(\omega_{22}, \omega_{23}) \in A^2; 2\omega_{22} = 2\omega_{23} = 0\}.$$

So we obtain

$$H_1(\mathcal{E}_2; H_1(\Sigma_2)) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

This isomorphism follows from the short exact sequence

$$0 \rightarrow \text{Ext}(H_0(\mathcal{E}_2; H_1(\Sigma_2)), A) \rightarrow H^1(\mathcal{E}_2; H_A) \rightarrow \text{Hom}(H_1(\mathcal{E}_2; H_1(\Sigma_2)), A) \rightarrow 0$$

since we have $H_0(\mathcal{E}_2; H_1(\Sigma_2)) = H_1(\Sigma_2)_{\mathcal{E}_2} = 0$ by Lemma 2.1. we complete the proof of Theorem 1.1.

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(Akira Kanada) Department of Mathematics, Tokyo Institute of Technology, Oh-okayama, Meguro, Tokyo 152-8551, Japan
 E-mail address: kanada.a.aa@m.titech.ac.jp